

A SHARP LOWER BOUND FOR SOME NEUMANN EIGENVALUES OF THE HERMITE OPERATOR

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ABSTRACT. This paper deals with the Neumann eigenvalue problem for the Hermite operator defined in a convex, possibly unbounded, planar domain Ω , having one axis of symmetry passing through the origin. We prove a sharp lower bound for the first eigenvalue $\mu_1^{odd}(\Omega)$ with an associated eigenfunction odd with respect to the axis of symmetry. Such an estimate involves the first eigenvalue of the corresponding one-dimensional problem. As an immediate consequence, in the class of domains for which $\mu_1(\Omega) = \mu_1^{odd}(\Omega)$, we get an explicit lower bound for the difference between $\mu(\Omega)$ and the first Neumann eigenvalue of any strip.

1. INTRODUCTION

In this paper we study the following Neumann eigenvalue problem

$$(1.1) \quad \begin{cases} -\operatorname{div} \left(\exp \left(-\frac{x^2+y^2}{2} \right) \nabla u \right) = \mu \exp \left(-\frac{x^2+y^2}{2} \right) u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a possibly unbounded, smooth domain in \mathbb{R}^2 and ν is the outward normal to $\partial\Omega$. As in the case of the Neumann Laplacian it is easily seen that the lowest eigenvalue of problem (1.1) is zero, the eigenfunction being any constant. Eigenfunctions u corresponding to higher eigenvalues must satisfy the orthogonality condition

$$\int_{\Omega} u \, d\gamma_2 = 0,$$

where $d\gamma_2$ stands for the standard normal Gaussian measure, that is

$$d\gamma_2 = d\gamma_x \otimes d\gamma_y = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{x^2}{2} \right) dx \otimes \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{y^2}{2} \right) dy.$$

Clearly the equation in (1.1) can be rewritten as follows

$$-\Delta u + x \cdot \nabla u = \mu u,$$

where at the left hand side the classical Hermite operator appears. When $\Omega = \mathbb{R}^2$, all the eigenvalues of (1.1) are known and corresponding eigenfunctions are the Hermite polynomials (see, e.g., [9]). When $\Omega \subsetneq \mathbb{R}^2$, much more less is known about the spectral properties of this operator.

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In the case of Dirichlet homogeneous boundary condition a Faber-Krahn type inequality has been established by Ehrhard in 1986. In [10] (see also [5]) he actually proved that, among all domains in \mathbb{R}^N having prescribed Gaussian measure, the half-space achieves the smallest eigenvalue. A sharp inequality concerning the ratio between the first two eigenvalues (the so-called PPW estimate) is contained in [4].

When a Neumann homogeneous boundary condition is prescribed, in [8] a Szegő-Weinberger type inequality is derived; more precisely the authors proved that

$$\mu_1(\Omega) \leq \mu_1(\Omega^\sharp),$$

where Ω is a smooth domain of \mathbb{R}^N , symmetric with respect to the origin and possibly unbounded, and Ω^\sharp is the ball, centered at the origin, with the same Gaussian measure as Ω .

In this paper we consider the class of planar convex sets having an axis of symmetry through the origin, say for instance the y -axis, and we denote by $\mu_1^{odd}(\Omega)$ the lowest Neumann eigenvalue with a corresponding eigenfunction odd with respect to the axis of symmetry. We prove a lower bound for $\mu_1^{odd}(\Omega)$ in the same spirit of the celebrated Payne-Weinberger estimate for the first nontrivial eigenvalue $\mu_1^\Delta(\Omega)$ of the Laplacian. In [14] (see also [3]) the authors proved that if Ω is a bounded convex domain in \mathbb{R}^N with diameter d , then

$$(1.2) \quad \mu_1^\Delta(\Omega) \geq \frac{\pi^2}{d^2}.$$

This result is asymptotically sharp, since π^2/d^2 is the first nontrivial Neumann eigenvalue of the one-dimensional Laplacian in $(-d/2, d/2)$. Instead of π^2/d^2 our estimate for $\mu_1^{odd}(\Omega)$ involves the first nontrivial eigenvalue $\mu_1(a, b)$ of the following one-dimensional problem

$$\begin{cases} -v'' + xv' = \mu v & \text{in } (a, b) \\ v'(a) = v'(b) = 0, \end{cases}$$

where $-\infty \leq a < b \leq \infty$ are related to the geometry of Ω . As well-known the following variational characterization holds

$$(1.3) \quad \mu_1(a, b) = \min_{\int_a^b z \, d\gamma_x = 0} \frac{\int_a^b (z')^2 \, d\gamma_x}{\int_a^b z^2 \, d\gamma_x}.$$

Our first main result is the following.

Theorem 1.1. *Let Ω be a convex, bounded domain in \mathbb{R}^2 , symmetric with respect to the y -axis. Let $a > 0$ and let $p, -q : (-a, a) \rightarrow \mathbb{R}$ be concave, even functions such that*

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -a < x < a, q(x) < y < p(x)\};$$

then

$$(1.4) \quad \mu_1^{odd}(\Omega) \geq \mu_1(-a, a),$$

where $\mu_1(-a, a)$ is defined in (1.3). Equality sign holds in (1.4) for every rectangle $(-a, a) \times (-b, b)$ with $b > 0$.

The case of unbounded sets is addressed in the next theorem. We will implicitly suppose that Ω is not a vertical strip, being this case trivial.

Theorem 1.2. *Let Ω be a C^2 , convex, unbounded domain in \mathbb{R}^2 , symmetric with respect to the y -axis. Let $a \in (0, +\infty]$ and let $p : (-a, a) \rightarrow \mathbb{R}$ be a concave, even, C^2 function such that*

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -a < x < a, y < p(x)\}.$$

Assume that Ω satisfies a uniform interior sphere condition. Then (1.4) holds true. Equality sign holds in (1.4) for every domain of the type $(-a, a) \times (-\infty, b)$ with $b \in \mathbb{R}$.

As for the classical Neumann Laplacian, the convexity assumption cannot be removed. Indeed, consider a set Ω_ϵ consisting of two squares connected by a corridor having width ϵ in such a way that the set is symmetric with respect to the y -axis. It is easy to verify that, as ϵ goes to zero, $\mu_1^{odd}(\Omega_\epsilon)$ goes to zero.

Very recently we found the paper [2] where the authors, among other things, prove that if Ω is a bounded convex subset of \mathbb{R}^n ($n \geq 2$), then

$$(1.5) \quad \mu_1(\Omega) \geq \mu_1\left(-\frac{d}{2}, \frac{d}{2}\right),$$

where d denotes the diameter of Ω (see also [13], [1]). Estimate (1.5) cannot be compared with ours (1.4) which involves $\mu_1^{odd}(\Omega)$, the maximum distance a of $\partial\Omega$ from the axis of symmetry, and holds true also for unbounded domains. In a forthcoming paper [6] we will prove that it is possible to “pass to the limit” in (1.5) as $d \rightarrow \infty$ and hence prove that when Ω is a convex, unbounded domain of \mathbb{R}^n ($n \geq 2$), then

$$\mu_1(\Omega) \geq \mu_1(\mathbb{R}) = 1.$$

2. THE CASE OF BOUNDED DOMAINS

In order to prove Theorem 1.1 we first decompose the domain Ω into convex subdomains, symmetric with respect to the y -axis, having small Gaussian measure and width. Then we prove a lower bound for a class of Sturm-Liouville problems, which is somehow a stronger version of the one-dimensional analogue to (1.4). Finally we use the boundedness of an eigenfunction corresponding to $\mu_1^{odd}(\Omega)$, together with its first and second derivatives, to pass from one to two dimensions.

Proof of Theorem 1.1. Step 1: Decomposition in horizontal strips. By approximation arguments (see, for example, [7] and also [12] p. 35), we may always assume $p, q \in C^2(-a, a)$ and that there exist $c_0, c_1 > 0$ such that

$$(2.1) \quad c_0 \leq |p'(x)|, |q'(x)| \leq c_1$$

for every $x \in (-a, a)$. Let us consider the set of straight-lines parallel to the x -axis and contained in the half-plane $\{y > p(a)\}$; one of them divides $\Omega^+ = \Omega \cap \{y > p(a)\}$ into two convex subdomains with the same Gaussian measure over each of which u has zero mean value with respect to $d\gamma_2$. Repeating this process n times we get

$$\gamma_2(\Omega_k) = \frac{\gamma_2(\Omega^+)}{2^n}, \quad \Omega^+ = \bigcup_{k=1}^n \Omega_k, \quad \Omega_k \text{ convex}, \quad \int_{\Omega_k} u d\gamma_2 = 0.$$

Clearly, by construction,

$$\Omega_k = \{(x, y) \in \mathbb{R}^2 : -a_k \leq x \leq a_k, d_k \leq y \leq p_k(x)\}.$$

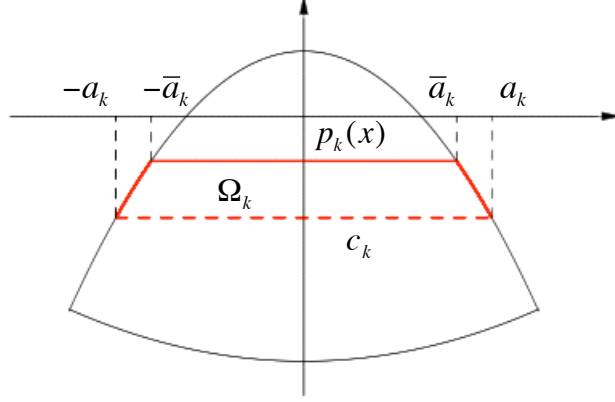


FIGURE 1.

Let us fix $\epsilon \in (0, \bar{\epsilon})$, where the value of $\bar{\epsilon}$ will be specified later; for sufficiently large n , we have that

$$(2.2) \quad \gamma_2(\Omega_k) < \epsilon \quad \text{and} \quad p_k(x) - d_k < \epsilon, \quad 1 \leq k \leq n.$$

Step 2: A one-dimensional auxiliary problem. Set

$$\phi_k(x) = \int_{d_k}^{p_k(x)} d\gamma_y;$$

because of the concavity of p and recalling the definition of p_k and d_k in Step 1, we have that $\phi_k(x)$ is a convex function. Let

$$(2.3) \quad \bar{\lambda}_k = \min \left\{ \frac{\int_{-a_k}^{a_k} (z')^2 \phi_k d\gamma_x}{\int_{-a_k}^{a_k} z^2 \phi_k d\gamma_x} : z \in C^1(-a_k, a_k), \int_{-a_k}^{a_k} z \phi_k d\gamma_x = 0 \right\}.$$

Then a function v_k which realizes the minimum in (2.3) must satisfy the condition

$$(2.4) \quad \int_{-a_k}^{a_k} v_k \phi_k d\gamma_x = 0$$

and the following Sturm-Liouville problem

$$\begin{cases} -[v'_k \phi_k \gamma_x]' = \bar{\lambda}_k v_k \phi_k \gamma_x & \text{in } (-a_k, a_k) \\ v'_k(-a_k) \phi_k(-a_k) = v'_k(a_k) \phi_k(a_k) = 0. \end{cases}$$

We differentiate with respect to x and introduce the new variable $w_k = v'_k \phi_k^{1/2}$. The function w_k satisfies the following problem

$$\begin{cases} -w_k'' + xw_k' + w_k \left[-\frac{1}{2} \frac{\phi_k''}{\phi_k} + \frac{3}{4} \left(\frac{\phi_k'}{\phi_k} \right)^2 - \frac{1}{2} x \frac{\phi_k'}{\phi_k} \right] = (\bar{\lambda}_k - 1)w_k & \text{in } (-a_k, a_k) \\ w(-a_k) = w(a_k) = 0. \end{cases}$$

Multiplying by $w_k(x) \exp\left(-\frac{x^2}{2}\right)$ and integrating over $(-a_k, a_k)$ we get

$$\begin{aligned} I_1 + I_2 &= \int_{-a_k}^{a_k} (w_k')^2 d\gamma_x + \int_{-a_k}^{a_k} \left(\frac{w_k}{\phi_k} \right)^2 \left[-\frac{1}{2} \phi_k \phi_k'' + \frac{3}{4} (\phi_k')^2 - \frac{1}{2} x \phi_k \phi_k' \right] d\gamma_x \\ &= (\bar{\lambda}_k - 1) \int_{-a_k}^{a_k} w_k^2 d\gamma_x. \end{aligned}$$

By construction, p_k is concave in $(-a_k, a_k)$ and constant in $(-\bar{a}_k, \bar{a}_k)$; then, recalling the definition of ϕ_k , we have

$$\begin{aligned} I_2 &= \int_{\bar{a}_k}^{a_k} \left(\frac{w_k}{\phi_k} \right)^2 \left[\phi_k e^{-\frac{p_k^2}{2}} \left(-p_k'' + p_k (p_k')^2 - x p_k' e^{-\frac{p_k^2}{2}} \right) + \frac{3}{2} (p_k')^2 e^{-p_k^2} \right] d\gamma_x \\ &\geq \int_{\bar{a}_k}^{a_k} \left(\frac{w_k}{\phi_k} \right)^2 \left[\phi_k e^{-\frac{p_k^2}{2}} \left(p_k (p_k')^2 - x p_k' e^{-\frac{p_k^2}{2}} \right) + \frac{3}{2} (p_k')^2 e^{-p_k^2} \right] d\gamma_x. \end{aligned}$$

It can be easily seen that there exists a positive constant L independent of n and k such that

$$\left| p_k(x) (p_k'(x))^2 - x p_k'(x) e^{-\frac{p_k(x)^2}{2}} \right| \leq L \quad \forall x \in (\bar{a}_k, a_k);$$

then, choosing $\bar{\epsilon} = \frac{3}{2} \frac{c_0^2}{L}$, by (2.1) and (2.2), the integral I_2 is bounded from below by a positive constant. Therefore

$$\bar{\lambda}_k - 1 \geq \frac{\int_{-a_k}^{a_k} (w_k')^2 d\gamma_x}{\int_{-a_k}^{a_k} w_k^2 d\gamma_x} \geq \lambda_1(-a_k, a_k),$$

where $\lambda_1(-a_k, a_k)$ is the first Dirichlet eigenvalue of the one-dimensional Hermite operator in $(-a_k, a_k)$. Since it can be easily seen that

$$(2.5) \quad \mu_1(-a_k, a_k) = \lambda_1(-a_k, a_k) + 1 \geq \lambda_1(-a, a) + 1 = \mu_1(-a, a),$$

we get

$$(2.6) \quad \bar{\lambda}_k \geq \mu_1(-a, a).$$

Step 3: Estimates by a dimension reduction process. Let u be an eigenfunction corresponding to $\mu_1^{odd}(\Omega)$ and let $M > 0$ be an upper bound for the absolute values of u and its first

and second derivatives. By Lagrange theorem we get

$$(2.7) \quad \left| \int_{\Omega_k} \left(\frac{\partial u}{\partial x} \right)^2 d\gamma_2 - \int_{-a_k}^{a_k} \left(\frac{\partial u}{\partial x}(x, d_k) \right)^2 \phi_k(x) d\gamma_x \right| \\ \leq 2M^2 \int_{-a_k}^{a_k} d\gamma_x \left(\int_{d_k}^{p_k(x)} (y - d_k) d\gamma_y \right) \leq 2M^2 \gamma_2(\Omega_k) \epsilon.$$

Analogously

$$(2.8) \quad \left| \int_{\Omega_k} u^2 d\gamma_2 - \int_{-a_k}^{a_k} u(x, d_k)^2 \phi_k(x) d\gamma_x \right| \leq 2M^2 \gamma_2(\Omega_k) \epsilon$$

and

$$(2.9) \quad \left| \int_{\Omega_k} u d\gamma_2 - \int_{-a_k}^{a_k} u(x, d_k) \phi_k(x) d\gamma_x \right| \leq M \gamma_2(\Omega_k) \epsilon.$$

Since the function $u(x, c_k) - \frac{1}{\gamma_2(\Omega_k)} \int_{-a_k}^{a_k} u(x, c_k) \phi_k(x) d\gamma_x$ satisfies condition (2.4), from (2.6), (2.7), (2.8), (2.9) and (2.5) we deduce

$$\begin{aligned} \int_{\Omega_k} |\nabla u|^2 d\gamma_2 &\geq \int_{\Omega_k} \left(\frac{\partial u}{\partial x} \right)^2 d\gamma_2 \\ &\geq \int_{-a_k}^{a_k} \left(\frac{\partial u}{\partial x}(x, c_k) \right)^2 \phi_k(x) d\gamma_x - 2M^2 \epsilon \gamma_2(\Omega_k) \\ &\geq \mu_1(-a_k, a_k) \int_{-a_k}^{a_k} \left[u(x, c_k) - \frac{1}{\gamma_2(\Omega_k)} \int_{-a_k}^{a_k} u(x, c_k) \phi_k(x) d\gamma_x \right]^2 \phi_k(x) d\gamma_x - 2M^2 \epsilon \gamma_2(\Omega_k) \\ &\geq \mu_1(-a, a) \left[\int_{-a_k}^{a_k} u(x, c_k)^2 \phi_k(x) d\gamma_x - \frac{1}{\gamma_2(\Omega_k)} \left(\int_{-a_k}^{a_k} u(x, c_k) \phi_k(x) d\gamma_x \right)^2 \right] - 2M^2 \epsilon \gamma_2(\Omega_k) \\ &\geq \mu_1(-a, a) \left[\int_{\Omega_k} u^2 d\gamma_2 - 2M^2 \epsilon \gamma_2(\Omega_k) - \frac{1}{\gamma_2(\Omega_k)} \left(\int_{\Omega_k} u d\gamma_2 + M \epsilon \gamma_2(\Omega_k) \right)^2 \right] - 2M^2 \epsilon \gamma_2(\Omega_k) \\ &= \mu_1(-a, a) \int_{\Omega_k} u^2 d\gamma_2 - \mu_1(-a, a) M^2 (2 + \epsilon) \epsilon \gamma_2(\Omega_k) - 2M^2 \epsilon \gamma_2(\Omega_k). \end{aligned}$$

Summing over k we get

$$\int_{\Omega^+} |\nabla u|^2 d\gamma_2 \geq \mu_1(-a, a) \int_{\Omega^+} u^2 d\gamma_2 - \mu_1(-a, a) M^2 (2 + \epsilon) \epsilon \gamma_2(\Omega^+) - 2M^2 \epsilon \gamma_2(\Omega^+).$$

Finally as ϵ goes to 0^+ we deduce

$$\int_{\Omega^+} |\nabla u|^2 d\gamma_2 \geq \mu_1(-a, a) \int_{\Omega^+} u^2 d\gamma_2.$$

Finally we can repeat the same arguments for $\Omega^- = \Omega \cap \{y < q(a)\}$ after a reflection about the x -axis, obtaining

$$\int_{\Omega^-} |\nabla u|^2 d\gamma_2 \geq \mu_1(-a, a) \int_{\Omega^-} u^2 d\gamma_2$$

and the thesis follows. \square

3. THE CASE OF UNBOUNDED DOMAINS

The arguments contained in the proof of Theorem 1.1 cannot be used to treat the case of unbounded domains, since in general an eigenfunction corresponding to μ_1^{odd} is not bounded. Consider, for instance, $\Omega = \{(x, y) \in \mathbb{R}^2 : y < 0\}$; then $\mu_1^{odd}(\Omega) = \mu_1(\Omega) = 1$ and a corresponding eigenfunction is $u(x, y) = x$. To overcome this difficulty we will consider a sequence of bounded sets Ω_n satisfying the assumptions of Theorem 1.1, invading Ω (see Figure 2) and then we pass to the limit in the eigenvalue estimate. A key point here is an extension result provided in Step 2 that could be of interest by its own. We cannot use the results already available in literature (see, for instance, [7], [11]), since we need a uniformly bounded sequence of extension operators $P_{\Omega_n} : H^1(\Omega_n, d\gamma_2) \rightarrow H^1(\mathbb{R}^2, d\gamma_2)$.

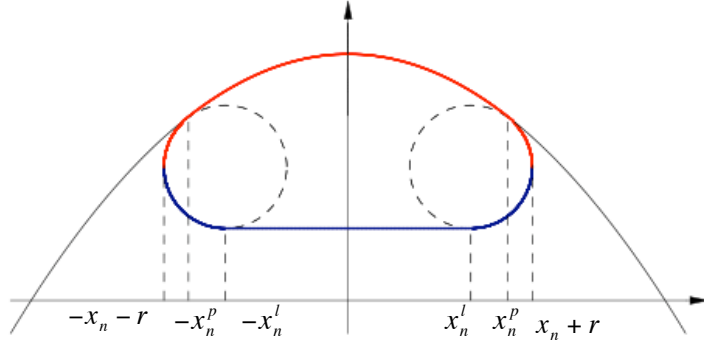


FIGURE 2.

Proof of Theorem 1.2. Step 1: A sequence of domains Ω_n invading Ω .

We distinguish two cases:

a) $\lim_{x \rightarrow a^-} p(x) = -\infty$;

b) $\lim_{x \rightarrow a^-} p(x) \in \mathbb{R}$.

Consider first case a). Let

$$G_n = \{(x, y) \in \Omega : y > -n\}, \quad n \geq \tilde{n} = [-p(0)] + 1,$$

where $[x]$ stands for the integer part of the real number x .

Note that $G_n \neq \emptyset$ for every $n \geq \tilde{n}$. In order to remove the wedges at the bottom of G_n , we consider, for every $n \geq \tilde{n}$, two equal disks D_n^\pm of radius r (whose value, independent of n , will be specified later) centered at $(\pm x_n, -n+r)$, contained in G_n , tangent both to $\partial\Omega$ and to $\{y = -n\}$ at the points $(\pm x_n^p, p(x_n^p))$ and $(\pm x_n^l, -n)$, respectively. Finally, let

$$p_n(x) = \begin{cases} p(x) & -x_n^l \leq x \leq x_n^p \\ -n + r + \sqrt{r^2 - (x + x_n)^2} & -x_n - r < x < -x_n^p \\ -n + r + \sqrt{r^2 - (x - x_n)^2} & x_n^p < x < x_n + r \end{cases}$$

and

$$q_n(x) = \begin{cases} -n & -x_n^p \leq x \leq x_n^l \\ -n + r - \sqrt{r^2 - (x + x_n)^2} & -x_n - r < x < -x_n^l \\ -n + r - \sqrt{r^2 - (x - x_n)^2} & x_n^l < x < x_n + r. \end{cases}$$

Set

$$\Omega_n = \{(x, y) \in \Omega : -x_n - r < x < x_n + r, q_n(x) < y < p_n(x)\}$$

(see Figure 2). Clearly Ω_n is a sequence of bounded, smooth, nested sets whose union coincides with Ω . Since Ω satisfies a uniform interior sphere condition we may choose

$$r = \tilde{r} = \frac{1}{2} \inf \frac{1}{|k(x, y)|} > 0,$$

where $k(x, y)$ stands for the curvature of $\partial\Omega$ at a generic point (x, y) . Let $\bar{\Omega}_n = \Omega_n + B(0, \tilde{r})$ be the exterior parallel set of Ω_n relative to $B(0, \tilde{r})$, that is the union of all closed disks of radius \tilde{r} whose centres lie in Ω_n . Denote by $\underline{\Omega}_n$ the interior parallel set of Ω_n relative to $B(0, \tilde{r})$, that is the union of the centres of all disks with radius \tilde{r} lying entirely in Ω_n .

Step 2: A uniformly bounded sequence of extension operators from $H^1(\Omega_n, d\gamma_2)$ onto $H^1(\mathbb{R}^2, d\gamma_2)$.

Let u_n be an eigenfunction corresponding to $\mu_1^{\text{odd}}(\Omega_n)$. We want to extend u_n to $\bar{\Omega}_n$ by reflection along the normal to $\partial\Omega_n$. Let $\Phi_n : (x, y) \in \Omega_n \setminus \underline{\Omega}_n \rightarrow (x_e, y_e) \in \bar{\Omega}_n \setminus \Omega_n$ be such a reflection. Φ_n is a one-to-one map and, denoted by

$$x_m = \frac{x + x_e}{2},$$

it holds

$$J_{\Phi_n}(x, y) = \frac{\partial(x_e, y_e)}{\partial(x, y)} = \begin{cases} \frac{1 + (p'_n(x_m))^2 + p''_n(x_m)(y - p_n(x_m))}{-1 - (p'_n(x_m))^2 + p''_n(x_m)(y - p_n(x_m))}, & y \geq -n + \tilde{r} \\ \frac{1 + (q'_n(x_m))^2 + q''_n(x_m)(y - q_n(x_m))}{-1 - (q'_n(x_m))^2 + q''_n(x_m)(y - q_n(x_m))}, & y < -n + \tilde{r}. \end{cases}$$

We explicitly observe that the quantities $p''_n(x_m)(y - p_n(x_m))$ and $q''_n(x_m)(y - q_n(x_m))$ are non-negative; so it is easy to verify that

$$|J_{\Phi_n}| \geq 1 \quad \text{in } \underline{\Omega}_n.$$

On the other hand, if $y \geq -n + \tilde{r}$,

$$p_n(x_m) - y = \frac{\bar{r}}{\sqrt{1 + (p'_n(x_m))^2}},$$

where

$$(3.1) \quad \bar{r} = \text{dist}((x, y), \partial\Omega) \in [0, \tilde{r}].$$

Then, from (3.1) we deduce that

$$0 \leq p''_n(x_m)(y - p_n(x_m)) \leq \frac{1}{2}(1 + (p'_n(x_m))^2).$$

Thus $|J_{\Phi_n}| \leq 3$ whenever $y \geq -n + \tilde{r}$.

Analogously one can treat the case $y < -n + \tilde{r}$ obtaining

$$(3.2) \quad 1 \leq |J_{\Phi_n}(x, y)| \leq 3, \quad \forall (x, y) \in \underline{\Omega}_n.$$

Define

$$(3.3) \quad \bar{u}_n(x_e, y_e) = u_n(\Phi_n^{-1}(x_e, y_e)) \quad \forall (x_e, y_e) \in \bar{\Omega}_n \setminus \Omega_n.$$

If $(x_e, y_e) = \Phi_n(x, y)$ with $y \geq -n + \tilde{r}$, we have

$$\begin{aligned} \frac{\partial \bar{u}_n(x_e, y_e)}{\partial x_e} &= \frac{\partial u_n(x, y)}{\partial x} \left(\frac{2}{1 + (p'_n(x_m))^2 + p''_n(x_m)(y - p_n(x_m))} - 1 \right) \\ &\quad + \frac{\partial u_n(x, y)}{\partial y} \frac{2p'_n(x_m)}{1 + (p'_n(x_m))^2 + p''_n(x_m)(y - p_n(x_m))} \\ \frac{\partial \bar{u}_n(x_e, y_e)}{\partial y_e} &= \frac{\partial u_n(x, y)}{\partial x} \frac{2p'_n(x_m)}{1 + (p'_n(x_m))^2 + p''_n(x_m)(y - p_n(x_m))} \\ &\quad + \frac{\partial u_n(x, y)}{\partial y} \left(\frac{2p'_n(x_m)^2}{1 + (p'_n(x_m))^2 + p''_n(x_m)(y - p_n(x_m))} - 1 \right). \end{aligned}$$

Analogous equalities hold whenever $y < -n + \tilde{r}$; then we can easily deduce that

$$(3.4) \quad \left(\frac{\partial \bar{u}_n(x_e, y_e)}{\partial x_e} \right)^2 + \left(\frac{\partial \bar{u}_n(x_e, y_e)}{\partial y_e} \right)^2 \leq 2 \left[\left(\frac{\partial u_n(x, y)}{\partial x} \right)^2 + \left(\frac{\partial u_n(x, y)}{\partial y} \right)^2 \right].$$

Now let $\theta_n \in C_0^\infty(\mathbb{R}^2)$ be such that $0 \leq \theta_n \leq 1$ in \mathbb{R}^2 , $\theta_n = 1$ on Ω_n , $\theta_n = 0$ in $\mathbb{R}^2 \setminus \bar{\Omega}_n$ and $|D\theta_n| \leq C$, with C independent of n and dependent only on r . Set

$$\tilde{u}_n = \theta_n \bar{u}_n.$$

To go on we claim that there exists a positive constant C , independent of n , such that

$$(3.5) \quad \exp \left(\frac{-\|\Phi_n(x, y)\|^2}{2} + \frac{x^2 + y^2}{2} \right) \leq C \quad \forall (x, y) \in \underline{\Omega}_n.$$

Indeed a straightforward computation yields that, if $y \geq -n + \tilde{r}$,

$$-\|\Phi_n(x, y)\|^2 + x^2 + y^2 = \frac{4\bar{r}}{\sqrt{1 + p'_n(x_m)^2}} (x_m p'_n(x_m) - p_n(x_m)),$$

where \bar{r} is defined in (3.1). The concavity of p_n in $[-x_n - \tilde{r}, x_n + \tilde{r}]$ ensures that

$$x_m p'_n(x_m) - p_n(x_m) \leq -p_n(0).$$

Hence, for $y \geq -n + \tilde{r}$, we have

$$\exp \left(\frac{-\|\Phi_n(x, y)\|^2}{2} + \frac{x^2 + y^2}{2} \right) \leq \exp \left(-\frac{2\bar{r}p_n(0)}{\sqrt{1 + p'_n(x_m)^2}} \right) \leq \max\{1, e^{-2rp(0)}\}.$$

Analogously, when $y < -n + \tilde{r}$, since, without loss of generality we may assume $-q_n(0) > 0$, we get

$$\exp \left(\frac{-\|\Phi_n(x, y)\|^2}{2} + \frac{x^2 + y^2}{2} \right) \leq 1$$

and the claim (3.5) follows.

Finally, by (3.2), (3.3) and (3.5) we get

$$\begin{aligned}
(3.6) \quad & \int_{\mathbb{R}^2} \tilde{u}_n^2 d\gamma_2 \\
&= \int_{\Omega_n} u_n^2 d\gamma_2 + \int_{\bar{\Omega}_n \setminus \Omega_n} \tilde{u}_n^2 d\gamma_2 \\
&= \int_{\Omega_n} u_n^2 d\gamma_2 + \int_{\Omega_n \setminus \underline{\Omega}_n} \theta_n^2(\Phi_n(x, y)) u_n^2(x, y) \exp\left(-\frac{\|\Phi_n(x, y)\|^2}{2} + \frac{x^2 + y^2}{2}\right) |J_{\Phi_n}| d\gamma_2 \\
&\leq C \int_{\Omega_n} u_n^2 d\gamma_2,
\end{aligned}$$

while (3.2), (3.4) and (3.6) imply

$$\begin{aligned}
& \int_{\mathbb{R}^2} |D\tilde{u}_n|^2 d\gamma_2 \\
&\leq C \int_{\Omega_n} u_n^2 d\gamma_2 + \int_{\Omega_n} |Du_n|^2 d\gamma_2 + C \int_{\Omega_n} |Du_n|^2 \exp\left(-\frac{\|\Phi_n(x, y)\|^2}{2} + \frac{x^2 + y^2}{2}\right) |J_{\Phi_n}| d\gamma_2 \\
&\leq C \left[\int_{\Omega_n} u_n^2 d\gamma_2 + \int_{\Omega_n} |Du_n|^2 d\gamma_2 \right].
\end{aligned}$$

We have proved that

$$(3.7) \quad \|\tilde{u}_n\|_{H^1(\mathbb{R}^2, d\gamma_2)} \leq C \|u_n\|_{H^1(\Omega_n, d\gamma_2)}$$

with C dependent only on \tilde{r} .

Remark 3.1. *Note that in Step 2 we do not use the symmetry assumption but just the fact that $\partial\Omega$ does intersect the y -axis. Therefore, by repeating the same arguments used in Step 2, we can prove that, if Ω is a convex, C^2 domain, satisfying a uniform interior sphere condition such that $\partial\Omega \cap \{x = 0\} \neq \emptyset$, then there exists an extension operator $P : H^1(\Omega, d\gamma_2) \rightarrow H^1(\mathbb{R}^2, d\gamma_2)$ such that*

$$(3.8) \quad \|Pu\|_{H^1(\mathbb{R}^2, d\gamma_2)} \leq C \|u\|_{H^1(\Omega, d\gamma_2)}$$

with C depending only on the radius of the interior sphere. In turn, as done for example in [11], from (3.8) we can derive the compactness of the embedding of $H^1(\Omega, d\gamma_2)$ into $L^2(\Omega, d\gamma_2)$.

Step 3: Passing to the limit.

Observe that the sequence $\mu_1^{odd}(\Omega_n)$ is bounded from above and from below by two positive constants. Indeed, using (1.4), we have

$$(3.9) \quad 1 = \mu_1(-\infty, +\infty) \leq \mu_1(-x_n - \tilde{r}, x_n + \tilde{r}) \leq \mu_1^{odd}(\Omega_n) \leq \frac{\gamma_2(\Omega_n)}{\int_{\Omega_n} x^2 d\gamma_2} \leq \frac{\gamma_2(\Omega_1)}{\int_{\Omega_1} x^2 d\gamma_2}.$$

Thus, up to a subsequence, $\mu_1^{odd}(\Omega_n)$ converge to a number $\mu > 0$.

Now, let us normalize u_n in such a way that $u_n(x, y) > 0$ if $x > 0$ and $\int_{\Omega_n} u_n^2 d\gamma_2 = 1$ for every n . From (3.9) and (3.7) we deduce that the sequence \tilde{u}_n is bounded in $H^1(\mathbb{R}^2, d\gamma_2)$. Then, up

to a subsequence, \tilde{u}_n converges to a function v weakly in $H^1(\Omega, d\gamma_2)$, strongly in $L^2(\Omega, d\gamma_2)$ and a.e. in Ω . Let $\varphi \in H^1(\Omega, d\gamma_2)$; then

$$\int_{\Omega} D\tilde{u}_n D\varphi d\gamma_2 = \int_{\Omega_n} Du_n D\varphi d\gamma_2 + \int_{\Omega \setminus \Omega_n} D\tilde{u}_n D\varphi d\gamma_2$$

and

$$\int_{\Omega} \tilde{u}_n \varphi d\gamma_2 = \int_{\Omega_n} u_n \varphi d\gamma_2 + \int_{\Omega \setminus \Omega_n} \tilde{u}_n \varphi d\gamma_2.$$

Since

$$\int_{\Omega \setminus \Omega_n} D\tilde{u}_n D\varphi d\gamma_2 \leq \left(\int_{\mathbb{R}^2} |D\tilde{u}_n|^2 d\gamma_2 \right)^{1/2} \left(\int_{\Omega \setminus \Omega_n} |D\varphi|^2 d\gamma_2 \right)^{1/2} \leq C \left(\int_{\Omega \setminus \Omega_n} |D\varphi|^2 d\gamma_2 \right)^{1/2}$$

and $\lim_{n \rightarrow \infty} \gamma_2(\Omega \setminus \Omega_n) = 0$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega \setminus \Omega_n} D\tilde{u}_n D\varphi d\gamma_2 = 0.$$

Analogously, it holds

$$\lim_{n \rightarrow \infty} \int_{\Omega \setminus \Omega_n} \tilde{u}_n \varphi d\gamma_2 = 0.$$

Then, up to subsequences,

$$\begin{aligned} \int_{\Omega} Dv D\varphi &= \lim_{n \rightarrow \infty} \int_{\Omega} D\tilde{u}_n D\varphi d\gamma_2 = \lim_{n \rightarrow \infty} \int_{\Omega_n} Du_n D\varphi d\gamma_2 \\ &= \lim_{n \rightarrow \infty} \mu_1^{odd}(\Omega_n) \int_{\Omega_n} u_n \varphi d\gamma_2 = \mu \lim_{n \rightarrow \infty} \int_{\Omega} \tilde{u}_n \varphi d\gamma_2 = \mu \int_{\Omega} v \varphi d\gamma_2. \end{aligned}$$

On the other hand, v inherits the sign of u_n , more precisely $v > 0$ if $x > 0$ and $v < 0$ if $x < 0$. Then v is an eigenfunction corresponding to $\mu_1^{odd}(\Omega)$. Finally, by applying (3.9), we have

$$\mu_1^{odd}(\Omega) = \lim_{n \rightarrow \infty} \mu_1^{odd}(\Omega_n) \geq \mu_1(-a, a)$$

and hence case a) is proved.

The proof of case b) is much more simple, since the boundary of Ω contains two parallel half-lines. Thus the reflection map through these two straight lines has obviously jacobian 1. \square

Finally, let $S_a = \{(x, y) \in \mathbb{R}^2 : |x| < a\}$. We observe that

$$\mu_1^{odd}(S_a) = \mu_1(-a, a) = 1 + \lambda_1(-a, a) = \mu_1(S_a) + \lambda_1(-a, a).$$

The following proposition is an immediate consequence of estimate (1.4).

Proposition 3.1. *If Ω satisfies assumptions of Theorem 1.1 or 1.2 and $\mu_1(\Omega) = \mu_1^{odd}(\Omega)$, then*

$$\mu_1(\Omega) - \mu_1(S_a) = \mu_1(\Omega) - 1 \geq \lambda_1(-a, a).$$

Remark 3.2. *As already observed in the Introduction, the equality*

$$(3.10) \quad \mu_1(\Omega) = \mu_1^{odd}(\Omega)$$

holds for instance when Ω is any disk centred at the origin or any square $(-l, l)^2$ ($l > 0$).

Anyway, all the assumptions of Theorem 1.1 or 1.2 are not enough to guarantee (3.10) to hold. Indeed, denoted by $T = (-1, 1) \times (-\infty, 0)$, it is easy to verify that $2 = \mu_1(T) < \mu_1^{odd}(T) = 3$ with corresponding eigenfunctions $u(x, y) = y^2 - 1$ and $u^{odd}(x, y) = x^3 - 3x$.

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